

Effects of multiple perturbation on solitary wave propagation and control

Jiuli Yin, Yanmin Wu, Lixin Tian, Wenxia Chen

Abstract—In this paper, we study the influence of complexity of external periodic perturbation on solitary waves of the mKdV equation. By using an improved Melnikov method, it is proved that the solitary wave always turns into chaotic state under arbitrary periodic perturbation. Furthermore, We observe that higher divergence (exponential separation of trajectories) in chaos phenomenon is resulted when the perturbation term possesses more frequencies. Corresponding to this phenomenon, we should apply larger control strength to suppress chaos which is excited from the system submitted to more perturbed frequencies. On the other hand, we find that the control strength are also affected by many factors: perturbed power, propagate speed and nonlinear parameter.

Index Terms— mKdV equation, multiple perturbation , chaos control, solitary waves

1 INTRODUCTION

The mKdV equation

$$u_t + au^2u_x + u_{xxx} = 0. \quad (1)$$

has many applications in nonlinear science and other fields. It can model Schottky barrier transmission lines and traffic congestion[1]. Many researches also focused on nonreciprocal propagation of solitary waves in granular chains with asymmetric potential barriers[2]. Ge et al [3] have derived the mKdV equations to describe the traffic jam near the critical points, and obtained the kink - antikink soliton solutions related to the traffic density waves. A systematic construction of Bäcklund transformation for the entire mKdV hierarchy is proposed, and solutions of these transformations for several cases are determined [4].

Melnikov method is an effective method to study the perturbed system. Chaos control of self-sustained electromechanical seismograph system is considered based on the Melnikov theory [8].

The Melnikov method is applied to non-smooth dynamical systems for studying the global behavior near a non-smooth homoclinic orbit under small time-periodic perturbations [9]. A bifurcation and chaos analysis of a magneto piezoelectric vibration energy harvester is presented by using Melnikov theory [10]. Identification and control of chaos in nonlinear gear dynamic systems is studied by Melnikov analysis [11].

It is noted that we observe the following facts: (1)The mKdV equation models many practical problems, so it is difficult to avoid the external perturbation. (2) External perturbation is complex and not simple, so it should be a combination of different perturbed frequencies. (3) Few studies on dynamical behavior of the mKdV equation with multiple perturbations is carried out.

Based on the above facts, we focus on studying the occurrence and control of chaos in the mKdV equation with multiple perturbations. Furthermore, we want to deeply understand the influence of the number of perturbed frequencies on the chaos occurrence and control.

The rest of the paper is organized as follows. In Section 2,

we study the effect of multiple perturbation on the propagation of solitary waves. In Section 3, we study the effects of multiple perturbation on controlled ability.

2 EFFECT OF MULTIPLE PERTURBATION ON SOLITARY WAVE PROPAGATION

2.1 Homoclinic orbits and solitary waves

As is well known, the solitary wave of equation (1) has the following form Supposing $u(x, t) = \alpha(x - ct)$ is a solitary wave of Eq. (1), we have

$$-c\alpha' + \alpha''' + a\alpha^2\alpha'' = 0, \quad (2)$$

where $\alpha'(x)$ is the derivative of $\alpha(x)$ w.r.t x .

Using the decay $\alpha(x)$ at infinity, we integrate (2) and obtain

$$-c\alpha + \alpha'' + \frac{a}{3}\alpha^3 = 0. \quad (3)$$

Eq. (3) makes sense for all $\alpha \in H_{loc}^1(\mathbb{R})$, and the following definition is therefore natural.

Definition 1. A function $\alpha \in H_{loc}^1(\mathbb{R})$ is a solitary wave solution to Eq.(1) if α satisfies (3) in distribution sense.

Define $\beta = \alpha'$, Eq. (3) can be expressed by

$$\begin{cases} \alpha' = \beta, \\ \beta' = c\alpha - \frac{a}{3}\alpha^3. \end{cases} \quad (4)$$

We know that solitary waves and have some contact with orbits, homoclinic orbit corresponds to a solitary wave solutions. Here we prove the existence of homoclinic orbits through bifurcation theory, and then to prove the existence of solitary waves.

Lemma 1. For any $c > 0$, system (4) admits two homoclinic

orbits associated with two solitary waves.

Proof. Firstly, we consider the specific equilibrium points of

system (4). It has three equilibrium points $E_1(-\sqrt{\frac{3c}{a}}, 0)$,

$E_2(\sqrt{\frac{3c}{a}}, 0)$ and $E_3(0, 0)$. Let J_{E_i} be the Jacobian matrix

for these equilibrium points and we obtain

$$J_{E_i} = \begin{bmatrix} 0 & 1 \\ c - a\alpha^2 & 0 \end{bmatrix}.$$

It is easy to find that its eigenvalues are $\lambda_{1,2}(J_{E_i}) = \pm\sqrt{-2c}$

and $\lambda_3(J_{E_3}) = \pm\sqrt{c}$. So the E_1, E_2 are the center equilibria.

and $E_3(0, 0)$ is a saddle point for any $c > 0$.

Secondly, system (4) has the following Hamiltonian function:

$$H(\alpha, \beta) = \beta^2 - c\alpha^2 + \frac{a}{6}\alpha^4 = h, \quad (5)$$

where h is a constant. It is noted that the Hamiltonian function is composed of two homoclinic orbits at the point E_3 . According to the bifurcation theory[12], system (4) has two solitary waves followed by two homoclinic orbits: Positive solitary

wave achieve its crest at $\alpha = \sqrt{\frac{6c}{a}}$, and negative solitary

wave has valley $\alpha = -\sqrt{\frac{6c}{a}}$.

For example, the corresponding homoclinic orbits and two solitary waves are shown in Fig. 1, In which the parameters are taken as $a = 6$ and $c = 3.8$.

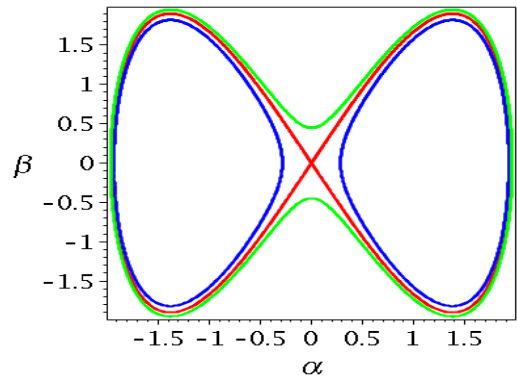


Fig. 1 (a)Phase portrait of system (4)

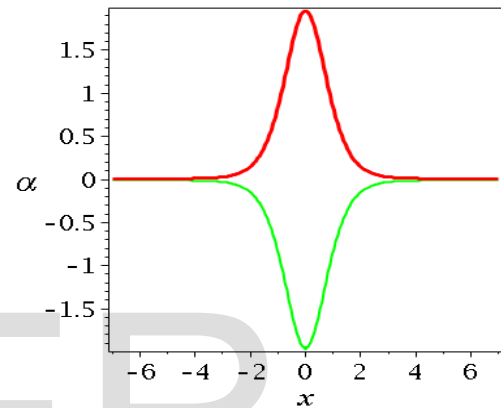


Fig. 1 (b)solitary wave. of system (4)

2.2 From solitary wave to chaos

When the perturbation $\varepsilon(x)$ exists, Eq. (3) becomes

$$-c\alpha + \alpha'' + \frac{a}{3}\alpha^3 = \varepsilon(x), \quad (6)$$

where $\varepsilon(x)$ is a general expression given by

$$\varepsilon(x) = \frac{d}{\sqrt{N}} \sum_{i=1}^N \cos(\omega_i x). \quad (7)$$

Here $\frac{d}{\sqrt{N}}$ and ω_i denote the amplitude and the frequency

respectively, in the perturbation signal function.

The reason for using Eq. (7) lie in two points: Firstly, the propagation of waves is subjected to the complicated external environmental perturbations, not a single frequency, so it should be multi-frequency. Secondly, The power of $\varepsilon(x)$

equals to $\frac{d^2}{2}$, independent of N . This allows us to focus our analysis to the richness of the frequency, while the signal power is kept as constant.

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Define $\beta = \alpha'$, system (6) can be expressed by

$$\begin{cases} \alpha' = \beta, \\ \beta' = c\alpha - \frac{a}{3}\alpha^3 + \frac{d}{\sqrt{N}} \sum_{i=1}^N \cos(\omega_i x). \end{cases} \quad (8)$$

Theorem 1. Given that $c > 0$, solitary waves in Eq. (3) always turn into chaos under the perturbation $\varepsilon(x)$ for any positive integer N .

Proof. Consider that $\frac{d}{\sqrt{N}}$ is a small perturbed parameter, $\omega_i (i=1,2,\dots,N)$ are some rational numbers, and the unperturbed homoclinic orbits are written as $(\alpha, \beta) = (\alpha_0(x), \beta_0(x))$. We can prove the existence of chaos by the Melnikov's Theorem. Define a Melnikov function for system (8) as:

$$\begin{aligned} M(x_0) &= \frac{d}{\sqrt{N}} \int_{-\infty}^{+\infty} \beta_0(x) \sum_{i=1}^N \cos(\omega_i(x+x_0)) dx \\ &= \frac{d}{\sqrt{N}} \sum_{i=1}^N \left[\int_{-\infty}^{+\infty} \beta_0(x) \cos(\omega_i x_0) \cos(\omega_i x) \right. \\ &\quad \left. - \beta_0(x) \sin(\omega_i x_0) \sin(\omega_i x) \right] dx. \end{aligned} \quad (9)$$

Since $\cos(\omega_i x)$ is even, we have

$$\int_{-\infty}^{+\infty} \beta_0(x) \cos(\omega_i x_0) \cos(\omega_i x) dx = 0 \text{ and (9) becomes}$$

$$\begin{aligned} M(x_0) &= -\frac{d}{\sqrt{N}} \sum_{i=1}^N \int_{-\infty}^{+\infty} \beta_0(x) \sin(\omega_i x_0) \sin(\omega_i x) dx \\ &= -\frac{2d}{\sqrt{N}} \sum_{i=1}^N \sin(\omega_i x_0) I_i, \end{aligned} \quad (10)$$

where $I_i = \int_0^{\infty} \beta_0(x) \sin(\omega_i x) dx$ is a function of ω_i .

Using the previous results and Melnikov's theorem [12,13], the stable and unstable manifolds intersect transversely, and the chaos occurs if $M(x_0) = 0$ and $M'(x_0) \neq 0$ for some x_0 .

Next we find some x_0 satisfying $\sin(\omega_i x_0) = 0$ and $\cos(\omega_i x_0) \neq 0$, (11)

for any ω_i . If such x_0 exists, we can prove chaos occurs.

For any positive integer n , $x_0 = \frac{n\pi}{\omega_1}$ satisfies $\sin(\omega_1 x_0) = 0$ and $\cos(\omega_i x_0) \neq 0$. When $i > 1$, we get

$\omega_i x_0 = \frac{\omega_i}{\omega_1} n\pi = \frac{p_i}{q_i} n\pi$, where p_i and q_i are positive integers. If we choose n as common multiple of $q_i (i=2,3,\dots,N)$, we find such x_0 satisfying (11) and the proof is finished.

To verify Theorem 1, we will investigate the Lyapunov exponents and phase portraits of system (8). Parameters of system (8) used for simulations are listed as follows: $a = 6$, $c = 3.8$, $w_1 = 0.05$, $w_2 = 0.53$, $w_3 = 0.2$, $w_4 = 0.42$, $w_5 = 0.61$, $w_6 = 0.81$, $w_7 = 0.15$, $w_8 = 0.73$ and $w_9 = 0.37$. The phase portraits are shown in Fig. 3 and the Lyapunov exponents are shown in Fig. 2, It is clearly that chaos always appears for all given cases.

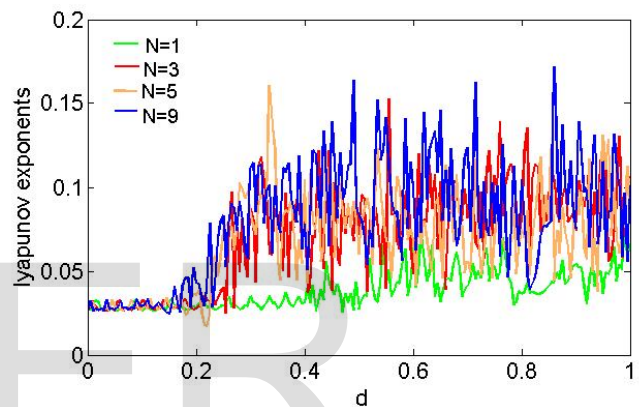


Fig. 2 Lyapunov exponents versus d with $N=1,3,5,9$.

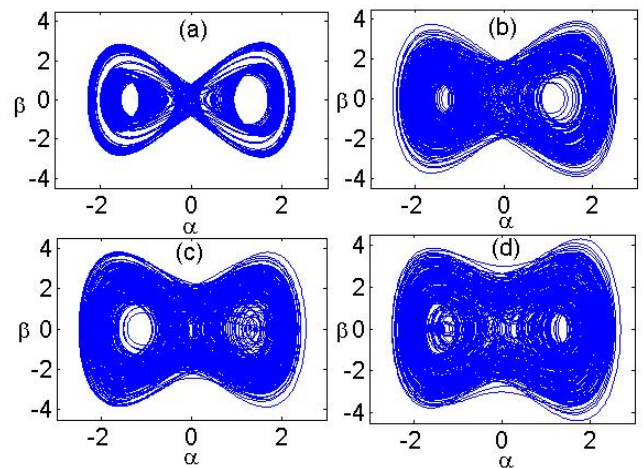


Fig. 3 The phase portraits of α and β for system (8): (a) $N = 1$, (b) $N = 3$, (c) $N = 5$, (d) $N = 9$.

Remark. We find that the complexity of perturbation affects the generation of chaos. As shown, Lyapunov exponents increase when N increase. Therefore, it implies that higher divergence (exponential separation of trajectories) is resulted when the perturbation of less frequency richness but same power is applied.

3 EFFECTS OF MULTIPLE PERTURBATION ON CONTROLLED SYSTEM

3.1 Melnikov analysis

we consider the chaos control of the perturbed system with a control term $\mu = e\beta$

$$\begin{cases} \alpha' = \beta, \\ \beta' = c\alpha - \frac{a}{2}\alpha^3 + \frac{d}{\sqrt{N}} \sum_{i=1}^N \cos(\omega_i x) + e\beta. \end{cases} \quad (12)$$

Next we measure how large the value of e should be set so that the perturbed system becomes non-chaotic. The Melnikov function of system (12) is given as below.

$$\begin{aligned} M(x_0) &= \int_{-\infty}^{+\infty} \beta_0(x) \left[e\beta_0(x) + \frac{d}{\sqrt{N}} \sum_{i=1}^N \cos \omega_i(x+x_0) \right] dx \\ &= e \int_{-\infty}^{+\infty} \beta_0^2(x) dx + \frac{d}{\sqrt{N}} \sum_{i=1}^N \left[\int_{-\infty}^{+\infty} \beta_0(x) \cos(\omega_i x_0) \cos(\omega_i x) \right. \\ &\quad \left. - \beta_0(x) \sin(\omega_i x_0) \sin(\omega_i x) \right] dx. \end{aligned}$$

Since $\cos(\omega_i x)$ is even, we have

$$\int_{-\infty}^{+\infty} \beta_0(x) \cos(\omega_i x_0) \cos(\omega_i x) dx = 0 \text{ and}$$

$$\begin{aligned} M(x_0) &= 2e \int_0^{\infty} \beta_0^2(x) dx - \frac{d}{\sqrt{N}} \sum_{i=1}^N \int_{-\infty}^{+\infty} \beta_0(x) \\ &\quad \times \sin(\omega_i x_0) \sin(\omega_i x) dx \\ &= 2eB - \frac{2d}{\sqrt{N}} \sum_{i=1}^N \sin(\omega_i x_0) I_i, \end{aligned}$$

where $I_i = \int_0^{\infty} \beta_0(x) \sin(\omega_i x) dx$ is a function of ω_i and

$$B = \int_0^{\infty} \beta_0^2(x) dx \text{ is a constant.}$$

If $M(x_0)$ has a simple zero and the corresponding critical parameter value is

$$(e)_0 = \frac{\left| \frac{d}{\sqrt{N}} \sum_{i=1}^n I_i \right|}{B}.$$

Following the method in [13], the general necessary condition for the existence of chaos can be derived as the facts:

(1) The homoclinic bifurcation occurs at $|e| = (e)_0$.

(2) Chaos still appears when $|e| < (e)_0$, which named uncontrollable region.

(3) Chaos can be controlled into a stable range when $|e| > (e)_0$, which named controllable region.

In order to deeply understand the effect of frequency richness (N) on chaos control, we will plot the curve of homoclinic bifurcation (namely $|e| = (e)_0$) which can help to find the uncontrollable region and controllable region.

We will take $N = 1, 3, 5, 9$ as examples, and the corresponding curves are green, red, yellow and blue. The fixed parameters are taken from the following data.

$a = 6, c = 3.8, \omega_1 = 0.05, \omega_2 = 0.53, \omega_3 = 0.2, \omega_4 = 0.42, \omega_5 = 0.61, \omega_6 = 0.81, \omega_7 = 0.15, \omega_8 = 0.73$ and $\omega_9 = 0.37$.

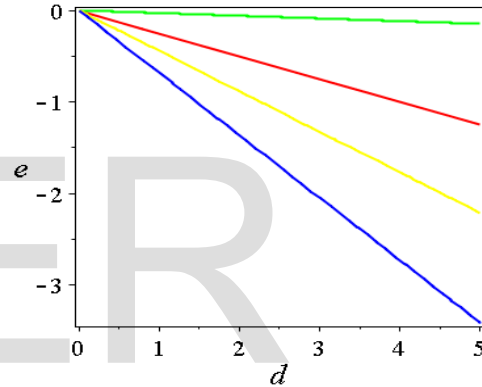


Fig.4 (a) Curve of homoclinic bifurcation ($e - d$) plane.

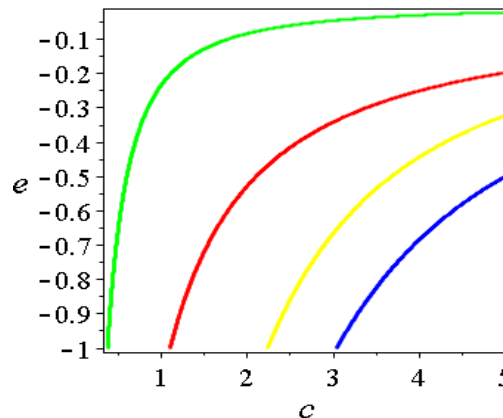


Fig.4 (b) Curve of homoclinic bifurcation ($e - c$) plane.

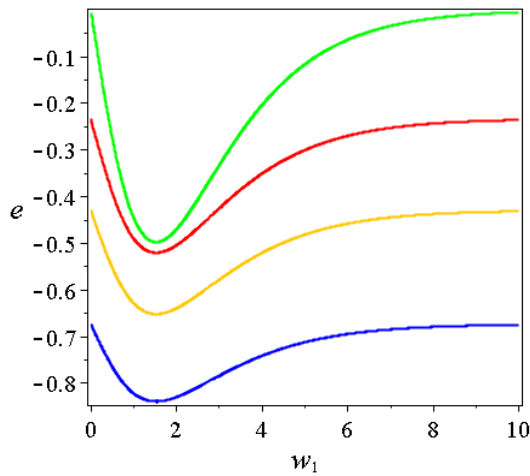


Fig.4 (c) Curve of homoclinic bifurcation ($e - \omega_1$) plane.

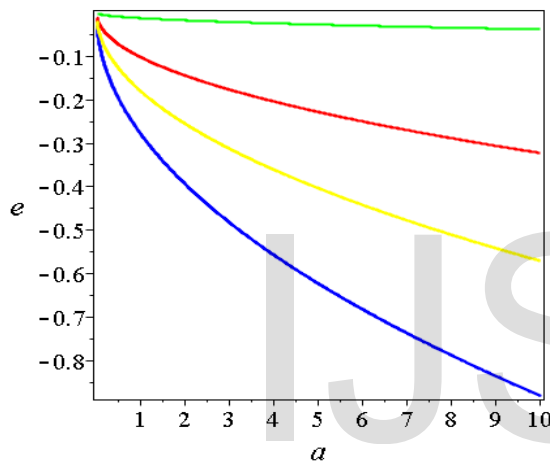


Fig.4 (d) Curve of homoclinic bifurcation ($e - a$) plane.

From Fig. 4, we find the following facts when N increases:

(1) The curve of homoclinic bifurcation downwards.

(2) The uncontrollable region widens and the controllable region narrows.

(3) It needs larger value of e to achieve chaos control when all other parameters are fixed.

We also find that the control strength (namely e) are affected by many factors. The larger value of e to achieve chaos control depends on increasing perturbed power $\frac{d^2}{2}$

control depends on increasing perturbed power $\frac{d^2}{2}$

Remark. From Fig. 5(c), it needs the maximum control strength when $\omega_1 \approx 1.7$, The frequency ω_1 near 1.7 has the greatest impact on the system and it is the most difficult to control, So we call this frequency awful frequency.

3.2 Lyapunov exponents analysis

Next the Lyapunov exponents and times series are used to study the effect of frequency richness on chaos control. The corresponding graphs are shown in Figs. 5-6. We find that it has similar phenomenon with the result of melnikov analysis:

a larger control parameter is needed to suppress chaos when N increases.

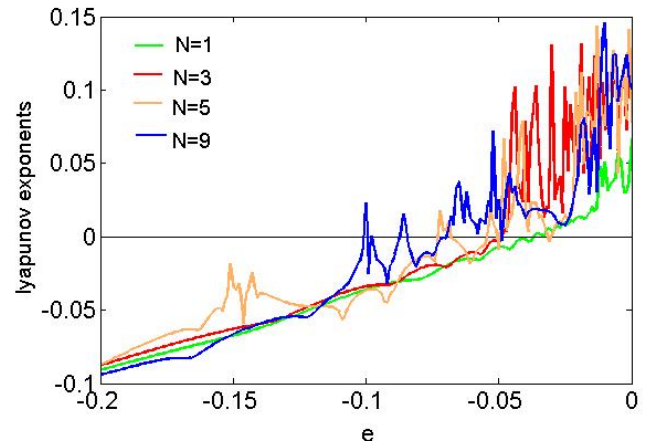


Fig. 5 Lyapunov exponents with different N .

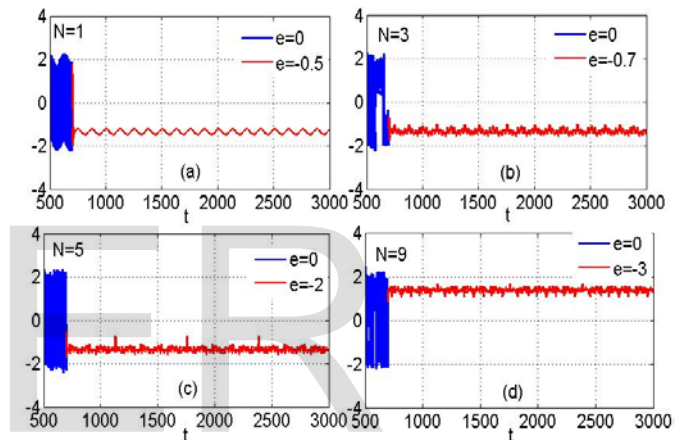


Fig. 6 The time series response for uncontrolled (blue) and controlled (red) system,

(a) $N = 1$, (b) $N = 3$, (c) $N = 5$, (d) $N = 9$.

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